

# GLOBAL EXISTENCE, UNIQUENESS AND STABILITY FOR NONLINEAR DISSIPATIVE SYSTEMS OF BULK-INTERFACE INTERACTION

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**ABSTRACT.** We consider a general class of nonlinear parabolic systems corresponding to thermodynamically consistent gradient structure models of bulk-interface interaction. The setting includes non-smooth geometries and e.g. slow, fast and “entropic” diffusion coefficients. The main results are global well-posedness and exponential stability of equilibria. As a part of the proof, we show bulk-interface maximum principles and a bulk-interface Poincaré inequality. The method of proof for global existence is a simple but very versatile combination of maximal parabolic regularity of the linearization, a priori  $L^\infty$ -bounds and a Schaefer fixed point argument. This allows us to extend the setting e.g. to Allen-Cahn dissipative dynamics and to include large classes of inhomogeneous boundary conditions and external forces.

## 1. INTRODUCTION

We consider a parabolic system of equations describing coupled bulk and interface dissipative processes of general quasi- and semilinear structure. This includes e.g. standard, slow and fast diffusion with Neumann boundary conditions and extends to Allen-Cahn- or chemical reaction-diffusion-type processes. The coupling of bulk and interface is of a general gradient structure first derived in [26], providing thermodynamical consistency. More precisely, we consider the following model equations.

**1.1. Model equations.** Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , be a bounded bulk domain with boundary  $\partial\Omega$ , divided into two open disjoint subdomains  $\Omega_+$  and  $\Omega_-$  by an interface  $\Gamma$  of dimension  $d-1$ . The scalar quantities  $u_+ : (0, T) \times \Omega_+ \rightarrow \mathbb{R}$ ,  $u_- : (0, T) \times \Omega_- \rightarrow \mathbb{R}$  and  $u_\Gamma : (0, T) \times \Gamma \rightarrow \mathbb{R}$  interact across the interface  $\Gamma$ , satisfying the evolution equations

$$\begin{cases} \dot{u}_+ - \operatorname{div}(k_+(u_+) \nabla u_+) &= f_+(u_+), & \text{in } (0, T) \times \Omega_+, \\ (k_+(u_+) \nabla u_+) \nu_+ + m_+(u)(u_+ - u_\Gamma) + m_\Gamma(u)(u_+ - u_-) &= g_+(u), & \text{on } (0, T) \times \Gamma, \\ (k_+(u_+) \nabla u_+) \nu_+ &= h_+(u_+), & \text{on } (0, T) \times \{\partial\Omega_+ \setminus \Gamma\}, \end{cases} \quad (1.1)$$

on the upper bulk part, and

$$\begin{cases} \dot{u}_- - \operatorname{div}(k_-(u_-) \nabla u_-) &= f_-(u_-), & \text{in } (0, T) \times \Omega_-, \\ (k_-(u_-) \nabla u_-) \nu_- + m_-(u)(u_- - u_\Gamma) + m_\Gamma(u)(u_- - u_+) &= g_-(u), & \text{on } (0, T) \times \Gamma, \\ (k_-(u_-) \nabla u_-) \nu_- &= h_-(u_-), & \text{on } (0, T) \times \{\partial\Omega_- \setminus \Gamma\}, \end{cases} \quad (1.2)$$

on the lower bulk part, coupled with the evolution

$$\begin{cases} \dot{u}_\Gamma - \operatorname{div}_\Gamma(k_\Gamma(u) \nabla_\Gamma u_\Gamma) - m_+(u)(u_+ - u_\Gamma) - m_-(u)(u_- - u_\Gamma) &= f_\Gamma(u), & \text{in } (0, T) \times \Gamma, \\ (k_\Gamma(u) \nabla_\Gamma u_\Gamma) \nu_\Gamma &= h_\Gamma(u_\Gamma), & \text{on } (0, T) \times \partial\Gamma, \end{cases} \quad (1.3)$$

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on  $\Gamma$ . Here,  $\nu_+, \nu_-, \nu_\Gamma$  denote the outer normal vector fields of  $\Omega_+$ ,  $\Omega_-$  and  $\Gamma$  and we use the shorthands

$$u = (u_+, u_-, u_\Gamma), \quad f = (f_+, f_-, f_\Gamma), \dots$$

The coefficient matrices  $k$ , the scalar transmission coefficients  $m$  and the external forces and inhomogeneous boundary conditions  $f, g, h$  may depend on the solution  $u$  and the space variables with

$$k_\pm : \Omega_\pm \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}^{d \times d} \quad \text{and} \quad k_\Gamma : \Gamma \times \mathbb{R}^3 \rightarrow \mathbb{R}_{\geq 0}^{(d-1) \times (d-1)}, \quad (1.4)$$

and

$$m_\Gamma, m_+, m_- : \Gamma \times \mathbb{R}^3 \rightarrow \mathbb{R}_{\geq 0}. \quad (1.5)$$

**1.2. Examples and applications.** Our main results on the system (1.1) – (1.3) are global well-posedness for particular choices of  $f, g, h$ , Theorem 3.1, and exponential stability in the case  $f, g, h = 0$ , Theorem 4.1. Our aim is to keep conditions on  $k$  and  $m$  very general, as long as they provide uniqueness, some regularity, positivity, and global stability of solutions, as these properties can be expected from a gradient flow structure. Particular examples of possible  $k_+$ , and likewise  $k_-, k_\Gamma$ , are  $k_+(u_+) = \kappa_0 u_+^{\rho-1}$  with constants  $\rho \in \mathbb{R}$  and  $\kappa_0 > 0$ , where the case  $k_+(u_+) = \frac{1}{u_+^2}$  is motivated by the entropic structure for the system in [26], cf. Subsection 5.1 and Corollary 5.1.

The transmission conditions  $m_\pm, m_\Gamma$  are assumed to be non-negative and it is sufficient that two of them are positive. Their dependence of  $u \in \mathbb{R}^3$  is assumed to be locally Lipschitz for uniqueness, including, for example, the case  $m_+(u) = \frac{1}{u_+^2 u_\Gamma}$ . They model a wide range of thermodynamically consistent reaction and adsorption processes between bulk and interfaces, motivated by the derivations in [26] and [21]. Again, we refer to Subsection 5.1 for details.

We have in mind several model application of the system (1.1) – (1.3):

- heat conduction within a bulk material, e.g. a semiconductor, separated into two parts by a thin active and heat conducting plate, e.g. made of metal. Particularly at high and low temperatures, thermal conductivity of these bulk and plate materials become non-linear in their dependence of temperature and non-equilibrium modeling of heat conduction across the plate leads to nonlinear transmission coefficients  $m$  of the type above, [36]. In technological applications, the geometry often includes sharp edges and singularities, e.g. where interface and boundary meet. A secondary aim of this work is thus to address these non-smooth settings in which standard regularity theory is not available.
- our results may provide insights for models of diffusion and transport of electrical charges in semiconductor devices, in particular, in three spatial dimensions, for which active interfaces often play a crucial role, cf. [21, 19].
- the quasi-linear structure of (1.1) – (1.3) may appear after a change of coordinates in free boundary problems, [33].
- the interaction of chemical species across bulk and interface is included in this model in a very general thermodynamically consistent form. We refer to [17] for the analysis of a particular model of this type with linear bulk diffusion in the bulk-surface situation  $\Omega_- = \emptyset$ . A related very active recent topic is the study of chemical reaction kinetics for catalysis including surfactants, cf. [3].
- we show global existence and uniqueness including semilinear  $f, g, h$ , as long as boundedness is preserved. This includes, for example, driving mechanisms of Allen-Cahn-type, cf. Corollary 3.6.

**1.3. Methods of proof and related work.** *Local* well-posedness of systems of type (1.1) – (1.3) was proved in [9], including also mixed boundary conditions. Here, we generalize the conditions on  $\Gamma$  and on the coefficients  $k, m$ . A specific catch of the local theory is that our geometric setting naturally leads to only Lipschitz regularity of  $\Omega_-$  and  $\Omega_+$ , as the separation of a smooth domain by an interface, even a plane, will usually create a kink. Optimal elliptic regularity for the system is thus not available in  $L^2$  or  $L^p$ . At the same time, using maximal parabolic regularity theory, an identification of the domain of the elliptic operator is a key property for proving well-posedness of the quasilinear system. The choice of a  $W^{-1,q}$ -setting with  $q > d$  for the abstract Cauchy problem associated to (1.1) – (1.3) in Section 2, turns out to work very well, providing both regularity and an elliptic isomorphy, cf. [10], and the flexibility to include inhomogenous Neumann boundary conditions in a natural way, Subsection 2.2. We refer to [8] for local-wellposedness of similar systems in an  $L^p$ -setting, where the dynamic surface covers the whole boundary of a smooth domain. An additional advantage of our functional analytic framework for (1.1) – (1.3) is that it has a very good perturbation theory. It is straightforward to include lower-order terms, time-dependence of coefficients and external forces, cf. Subsection 5.2. To obtain *global existence* of solutions, we use a Schaefer fixed point argument, combined with results on non-autonomous maximal parabolic regularity for the linear system and a maximum principle. Schaefer’s fixed point theorem is a standard tool in the treatment of quasilinear elliptic problems [18], but it seems to be seldomly used explicitly for parabolic problems, cf. [2]. Here, it is a simple but powerful tool for proving global existence, extending to regularity and uniqueness, and providing a method that may easily be adopted to (bulk) systems in the future. The proof of a maximum principle for system (1.1) – (1.3), Lemma 3.5, is quite elementary, but makes exact use of the gradient structure of the bulk-interface interaction terms. In addition, it is straightforward to extend the proof to other driving mechanisms like Allen-Cahn-type energies, cf. Corollary 3.6. We refer e.g. to [6] for recent results on Allen-Cahn equations with dynamic interface conditions. In contrast to the situation here, with an additional variable on the interface, dynamic interface/boundary conditions have been studied much more extensively. We refer e.g. to [35, 12, 28, 8] for recent results.

Under the assumption  $f, g, h = 0$ , we show *exponential stability* of equilibria of (1.1) – (1.3) under the constraint of mass conservation. The result follows from a bulk-interface Poincaré inequality,

$$\|u - u^\infty\|_{L^{2,2}}^2 \leq C(\|\nabla u\|_{L^{2,2}}^2 + \|u_+ - u_\Gamma\|_{L^2(\Gamma)}^2 + \|u_- - u_\Gamma\|_{L^2(\Gamma)}^2 + \|u_+ - u_-\|_{L^2(\Gamma)}^2),$$

in Lemma 4.2. Again, the proof is fairly straightforward, but it shows how the gradient structure proposed in the modeling of bulk-interface interaction in [26] can be adapted in the analysis and it shows that the coupling, in its generality, is sufficiently strong for pushing the system into global equilibrium.

**Outline.** The paper is organized as follows. In the next section, we fix basic assumptions on the geometry and coefficient functions in (1.1) – (1.3) and collect preliminary results on the bilinear form of energy dissipation associated to the system. In Section 3, the main result on existence and uniqueness of global weak solutions is proved, including a bulk-interface maximum principle. In Section 4, we show the bulk-interface Poincaré inequality, providing exponential stability for a global equilibrium under mass conservation. In the last Section 5, we discuss the relation of the model to the entropic Onsager system of heat diffusion and transfer derived in [26] and comment on straightforward extensions of the main results like the case of  $\Omega_- = \emptyset$ , higher regularity, dependence of coefficients on time, and the inclusion of lower-order perturbations.

## 2. BASIC ASSUMPTIONS AND ABSTRACT FRAMEWORK

The aim of this section is to define a functional analytic framework for equations (1.1) – (1.3) that works for very general geometric bulk-interface settings and transmission conditions. We introduce basic assumptions on the geometry of the domains  $\Omega_{\pm}, \Gamma$  and coefficient functions  $k, m$  that hold for the remainder of the paper. In Subsection 2.2, we construct a suitable linearization of the problem and in Subsection 2.3, we recall basic facts on maximal parabolic regularity, define suitable solution spaces and prove useful embedding results.

**2.1. Assumptions on geometry and coefficients.** Our assumptions on the geometry are quite general in the sense that only minimal smoothness is required and that bulk and interface need only interact locally and may touch non-smoothly.

**Assumption 2.1** (on  $\Omega_+, \Omega_-$  and  $\Gamma$ ). *The bulk domains  $\Omega_+$  and  $\Omega_-$  are bounded Lipschitz domains, cf. [22, Def. 1.2.12]. The interface  $\Gamma$  is a  $d - 1$ -dimensional  $C^1$ -manifold with Lipschitz boundary  $\partial\Gamma \subset \partial\Omega$ .*

We note that even if  $\Omega$  were smoother, at least one of the domains  $\Omega_+$  and  $\Omega_-$  should not be assumed to be smoother than Lipschitz as the division of  $\Omega$  by  $\Gamma$ , e.g. by a plane, will usually create a kink.

For  $q \in [1, \infty]$ ,  $L^q(\omega)$  denotes the usual real Lebesgue space of  $q$ -integrable functions on a domain or manifold  $\omega$ ,  $W^{m,q}(\omega)$  denote the usual  $L^q$ -Sobolev spaces of order  $m \in \mathbb{N}$  and  $C^\alpha(\omega)$  are the uniform Hölder spaces of exponent  $\alpha \geq 0$  with  $C^0(\omega) = C(\overline{\omega})$  if  $\omega$  is bounded. We introduce a convenient notation for function spaces related to (1.1) – (1.3). For  $q, q_\Gamma \in [1, \infty]$ , using  $\mathcal{H}_{d-1}$  the  $d - 1$ -dimensional Hausdorff measure on  $\Gamma$ , define

$$\begin{aligned} L^{q,q_\Gamma} &:= L^q(\Omega_+) \times L^q(\Omega_-) \times L^{q_\Gamma}(\Gamma), \\ W^{1,q,q_\Gamma} &:= W^{1,q}(\Omega_+) \times W^{1,q}(\Omega_-) \times W^{1,q_\Gamma}(\Gamma), \quad \text{and} \\ C^{\alpha,\alpha_\Gamma} &:= C^\alpha(\Omega_+) \times C^\alpha(\Omega_-) \times C^{\alpha_\Gamma}(\Gamma), \end{aligned}$$

where  $\alpha, \alpha_\Gamma \geq 0$ .

Note that since  $\Gamma$  is a smooth part of the boundary of  $\Omega_+$ , the trace operator

$$\text{tr}_\Gamma : W^{1,q}(\Omega_+) \rightarrow L^q(\Gamma) \tag{2.1}$$

is well-defined and continuous (likewise for  $\Omega_-$ ). We write

$$\text{tr}_\Gamma u = (\text{tr}_\Gamma u_+, \text{tr}_\Gamma u_-, u_\Gamma) \tag{2.2}$$

for the trace components of  $u$  on the interface  $\Gamma$ . Often, the operator  $\text{tr}_\Gamma$  is omitted in the notation for integrals and as, for example, in the statement of the model equations (1.3) on  $\Gamma$ .

With slight abuse of notation but consistent with notation for mixed boundary conditions (cf. [9]), dual Sobolev spaces are denoted by

$$W^{-1,q}(\omega) := (W^{1,q'}(\omega))' \quad \text{and} \quad W^{-1,q,q_\Gamma} := (W^{1,q',q'_\Gamma})'$$

with  $\frac{1}{q} + \frac{1}{q'} = \frac{1}{q_\Gamma} + \frac{1}{q'_\Gamma} = 1$ . Additionally, for  $-\infty < l \leq L < +\infty$  and  $n \in \mathbb{N}$ , let

$$(\mathbb{R}_l^L)^n := \{v \in \mathbb{R}^n : l \leq v_i \leq L \text{ for } i = 1, \dots, n\}, \quad \text{and}$$

$$C_l^L = \{u \in C^{0,0} : l \leq u_{\pm}(x), u_\Gamma(y) \leq L \text{ for all } x \in \Omega_{\pm}, y \in \Gamma\}.$$

**Assumption 2.2** (Assumptions on  $k$  and  $m$ ). *Let  $k$  and  $m$  be given as in (1.4) and (1.5) and let  $-\infty < l < L < +\infty$  be given constants.*

- (1) Uniformly in  $u \in (\mathbb{R}_l^L)^3$ , the coefficient matrices  $k(\cdot, u)$  are measurable, bounded and elliptic, i.e. there are constants  $\underline{k}, \bar{k} > 0$  such that

$$\|k(\cdot, u)\|_{L^\infty} \leq \bar{k}, \quad (2.3)$$

and such that for all  $x \in \mathbb{R}^d$ ,  $y \in \mathbb{R}^{d-1}$ ,

$$x \cdot k_\pm(\cdot, u)x \geq \underline{k}|x|^2 \quad \text{and} \quad y \cdot k_\Gamma(\cdot, u)y \geq \underline{k}|y|^2, \quad (2.4)$$

almost everywhere in  $\Omega_\pm$ ,  $\Gamma$ . In particular,  $\underline{k}$  and  $\bar{k}$  may depend on  $l, L$ , but not on  $u \in (\mathbb{R}_l^L)^3$ .

- (2) Uniformly in  $u \in (\mathbb{R}_l^L)^3$ , the transmission coefficients  $m_\pm, m_\Gamma$  are measurable and there are constants  $\underline{m}, \bar{m} > 0$  such that

$$\|m(\cdot, u)\|_{L^\infty(\Gamma)} \leq \bar{m} \quad (2.5)$$

and such that at least two of the three transmission functions, e.g.  $m_+, m_\Gamma$  are positively bounded from below,

$$\underline{m} \leq m_+(\cdot, u), m_\Gamma(\cdot, u), \quad (2.6)$$

and the third transmission function is non-negative,

$$0 \leq m_-(\cdot, u),$$

almost everywhere in  $\Gamma$ . Note that again,  $\underline{m}, \bar{m}$  may depend on  $l, L$ , but not on  $u \in (\mathbb{R}_l^L)^3$ .

- (3) The functions  $\mathbb{R} \ni u_\pm \mapsto k_\pm(x, u_\pm)$ ,  $\mathbb{R}^3 \ni u \mapsto k_\Gamma(y, u)$  and  $\mathbb{R}^3 \ni u \mapsto m(y, u)$  are locally Lipschitz uniformly in  $y \in \Gamma, x \in \Omega_\pm$ .
- (4) If  $d = 3$ , then  $k_\pm$  are of the form  $k_\pm(x, u_\pm) = \kappa_\pm(x, u_\pm)\varkappa_\pm(x)$ . The functions  $\kappa_\pm: \Omega_\pm \times \mathbb{R} \rightarrow \mathbb{R}$  are scalar, satisfy 2.2(3) and for all  $u_\pm \in \mathbb{R}$ , we have  $\kappa_\pm(\cdot, u_\pm) \in C^0(\Omega_\pm)$  with  $\underline{k} \leq \kappa_\pm(\cdot, u_\pm)$ . The functions  $\varkappa_\pm: \Omega_\pm \rightarrow \mathbb{R}_{\geq 0}^{3 \times 3}$  satisfy 2.2(1) and are uniformly continuous on  $\Omega_\pm$ .

For examples of coefficients  $k, m$  that satisfy Assumption 2.2, we refer e.g. to the introduction, Subsection 1.2 and to Section 5.1. In particular, the original modelling in [26, Sect. 4.2] is included as a special case. Note also that Assumption 2.2(4) may be relaxed, cf. Remark 5.5.

**2.2. A bilinear form and an elliptic operator associated to this problem.** The dissipation in (1.1) – (1.3) across  $\Gamma$  is governed by the transmission coefficient matrix  $\mathbf{m}$  given by

$$\mathbf{m} = \begin{pmatrix} m_+ + m_\Gamma & -m_\Gamma & -m_+ \\ -m_\Gamma & m_- + m_\Gamma & -m_- \\ -m_+ & -m_- & m_+ + m_- \end{pmatrix}.$$

Under Assumption 2.2(2),  $\mathbf{m}$  is positive semi-definite and for  $r = (r_+, r_-, r_\Gamma) \in \mathbb{R}^3$ ,

$$r \cdot \mathbf{m}r = 0 \text{ a.e., if and only if } r_+ = r_- = r_\Gamma. \quad (2.7)$$

Let  $-\infty < l \leq L < +\infty$ . For fixed  $u \in C_l^L$ , we define the bilinear form

$$\mathbf{a}_u: W^{1,2,2} \times W^{1,2,2} \rightarrow \mathbb{R}$$

by

$$\mathbf{a}_u(\psi, \varphi) := \mathbf{l}_u(\psi, \varphi) + \mathbf{m}_u(\psi, \varphi),$$

where

$$\begin{aligned} \mathbf{l}_u(\psi, \varphi) &:= \int_{\Omega_+} \nabla \psi_+ \cdot k_+(u_+) \nabla \varphi_+ \, dx + \int_{\Omega_-} \nabla \psi_- \cdot k_-(u_-) \nabla \varphi_- \, dx \\ &\quad + \int_\Gamma \nabla_\Gamma \psi_\Gamma \cdot k_\Gamma(u_\Gamma) \nabla_\Gamma \varphi_\Gamma \, dy, \\ &=: \mathbf{l}_{u,+}(\psi_+, \varphi_+) + \mathbf{l}_{u,-}(\psi_-, \varphi_-) + \mathbf{l}_{u,\Gamma}(\psi_\Gamma, \varphi_\Gamma), \end{aligned}$$

and

$$\mathbf{m}_u(\psi, \varphi) = \int_{\Gamma} \text{tr}_{\Gamma} \psi \cdot \mathbf{m} \text{tr}_{\Gamma} \varphi \, d\mathcal{H}_{d-1}.$$

By (2.1), the form  $\mathbf{a}_u$  is well-defined and continuous. Due to (2.7) and Assumption 2.2,

$$\mathbf{a}_u(\varphi, \varphi) \geq 0 \quad \text{and} \quad \mathbf{a}_u(\varphi, \varphi) = 0 \text{ if and only if } \varphi_+ = \varphi_- = \varphi_{\Gamma} \equiv \text{const.} \quad (2.8)$$

The form  $\mathbf{a}_u$  induces an operator  $\mathcal{A}_u : W^{1,2,2} \rightarrow W^{-1,2,2}$  by

$$\mathcal{A}_u(\psi)(\varphi) := \mathbf{a}_u(\psi, \varphi), \quad \text{for all } \psi, \varphi \in W^{1,2,2}.$$

For  $q, q_{\Gamma} \in [2, \infty)$ , let  $\mathcal{A}_u^{q, q_{\Gamma}}$  be the closed and densely defined restriction of  $\mathcal{A}_u$  to  $W^{-1, q, q_{\Gamma}}$ . We write  $\mathcal{L}_u^{q, q_{\Gamma}}$  for the divergence operator in  $W^{-1, q, q_{\Gamma}}$  analogously induced by  $\mathbf{l}_u$  and  $\mathcal{L}_{u,+}^{q, q_{\Gamma}}$ ,  $\mathcal{L}_{u,-}^{q, q_{\Gamma}}$  and  $\mathcal{L}_{u,\Gamma}^{q, q_{\Gamma}}$  for the Neumann operators induced by  $\mathbf{l}_{u,+}$ ,  $\mathbf{l}_{u,-}$  and  $\mathbf{l}_{u,\Gamma}$  on the domains  $\Omega_+$ ,  $\Omega_-$  and  $\Gamma$ , respectively. We write  $\mathcal{M}_u^{q, q_{\Gamma}}$  for the bounded transmission operator given by

$$\mathcal{M}_u^{q, q_{\Gamma}}(\psi)(\varphi) := \mathbf{m}_u(\psi, \varphi), \quad \psi \in \text{dom}(\mathcal{L}_u^{q, q_{\Gamma}}), \varphi \in W^{1, q', q'_{\Gamma}}, \quad (2.9)$$

so that

$$\mathcal{A}_u^{q, q_{\Gamma}} = \mathcal{L}_u^{q, q_{\Gamma}} + \mathcal{M}_u^{q, q_{\Gamma}}.$$

The external forces and inhomogeneous Neumann boundary conditions  $f, g, h$  in (1.1)–(1.3) are realized as a  $W^{-1, q, q_{\Gamma}}$ -functional  $\mathcal{F}(u)$  with components  $\mathcal{F}_+(u) \in W^{-1, q}(\Omega_+)$ ,  $\mathcal{F}_-(u) \in W^{-1, q}(\Omega_-)$  and  $\mathcal{F}_{\Gamma}(u) \in W^{-1, q_{\Gamma}}$  given by

$$\begin{aligned} \mathcal{F}_+(u)(\varphi_+) &= \int_{\Omega_+} f_+(u_+) \varphi_+ \, dx + \int_{\Gamma} g_+(u) \text{tr}_{\Gamma} \varphi_+ \, d\mathcal{H}_{d-1} + \int_{\partial\Omega_+ \setminus \Gamma} h_+(u_+) \text{tr}_{\partial\Omega_+ \setminus \Gamma} \varphi_+ \, d\mathcal{H}_{d-1} \\ \mathcal{F}_-(u)(\varphi_-) &= \int_{\Omega_-} f_-(u_-) \varphi_- \, dx + \int_{\Gamma} g_-(u) \text{tr}_{\Gamma} \varphi_- \, d\mathcal{H}_{d-1} + \int_{\partial\Omega_- \setminus \Gamma} h_-(u_-) \text{tr}_{\partial\Omega_- \setminus \Gamma} \varphi_- \, d\mathcal{H}_{d-1} \\ \mathcal{F}_{\Gamma}(u)(\varphi_{\Gamma}) &= \int_{\Gamma} f_{\Gamma}(u) \varphi_{\Gamma} \, d\mathcal{H}_{d-1} + \int_{\partial\Gamma} h_{\Gamma}(u_{\Gamma}) \text{tr}_{\partial\Gamma} \varphi_{\Gamma} \, d\mathcal{H}_{d-2}, \end{aligned}$$

for all  $\varphi \in W^{1, q', q'_{\Gamma}}$ . Using suitable trace embedding results, it follows that  $\mathcal{F}(u)$  is well-defined, if

$$\begin{aligned} f(u) &\in L^{p, p_{\Gamma}}, \quad \text{with } p > \frac{d}{d+1-\frac{d}{q}}, \text{ and } p_{\Gamma} > \frac{d-1}{d-\frac{d-1}{q_{\Gamma}}} \text{ if } d = 3, \\ &\text{or } p_{\Gamma} > 1 \text{ if } d = 2, \\ g_{\pm}(u), h_{\pm}(u) &\in L^{\rho}(\Gamma), \quad \text{with } \rho > \frac{d-1}{d-\frac{d}{q}}, \text{ and, if } d = 3, \\ h_{\Gamma}(u_{\Gamma}) &\in L^{\rho_{\Gamma}}(\partial\Gamma), \quad \text{with } \rho_{\Gamma} > 1. \end{aligned}$$

We interpret the set of equations (1.1), (1.2) and (1.3) as the quasilinear problem

$$\dot{u}(t) + \mathcal{A}_{u(t)} u(t) = \mathcal{F}(u(t)), \quad u(0) = u_0, \quad (2.10)$$

posed in  $W^{-1, q, q_{\Gamma}}$ ,  $q, q_{\Gamma} \geq 2$ .

**2.3. Maximal parabolic regularity and useful embeddings.** For  $T > 0$ , let in the following  $J_T = (0, T)$  be a bounded time interval. We briefly recall the notion of maximal  $L^r(J_T; X)$ -regularity for a Banach space  $X$ .

**Definition 2.3.** Let  $1 < r < \infty$ , let  $X$  be a Banach space and assume that  $B$  is a closed operator in  $X$  with dense domain  $\text{dom}(B) \subset X$ , equipped with the graph norm. We say that  $B$  satisfies maximal  $L^r(J_T; X)$ -regularity if for all  $u^0 \in (\text{dom}(B), X)_{1-\frac{1}{r}, r}$  and  $f \in L^r(0, T; X)$  there is a unique solution

$$u \in L^r(J_T; \text{dom}(B)) \cap W^{1, r}(J_T; X)$$

of the abstract Cauchy problem

$$\begin{cases} \dot{u} + Bu &= f, \\ u(0) &= u^0, \end{cases}$$



posed in  $X$ , satisfying

$$\|\dot{u}\|_{L^r(J_T; X)} + \|Bu\|_{L^r(J_T; X)} \leq C(\|u^0\|_{(\text{dom}(B), X)_{1-\frac{1}{r}, r}} + \|f\|_{L^r(J_T; X)})$$

with a constant  $C > 0$  independent of  $u^0$  and  $f$  (see e.g. [1, Ch. III.1]).

Note that the notion of maximal  $L^r(J_T; X)$ -regularity is actually independent of  $1 < r < \infty$  and  $T > 0$ , cf. [14]. In the following, for  $q, q_\Gamma \geq 2$ ,  $1 < r < \infty$  and given  $u \in C^{0,0}$ , we consider maximal regularity of  $\mathcal{A}_u^{q, q_\Gamma}$ , so we define

$$\text{MR}_{q, q_\Gamma}^r := L^r(J_T; \text{dom}(\mathcal{A}_u^{q, q_\Gamma})) \cap W^{1, r}(J_T; W^{-1, q, q_\Gamma})$$

as the corresponding solution space and

$$X_{q, q_\Gamma}^r := (\text{dom}(\mathcal{A}_u^{q, q_\Gamma}), W^{-1, q, q_\Gamma})_{1-\frac{1}{r}, r}$$

as the corresponding time trace space.

In Lemma 3.2 below, we show that there are  $q > d$ ,  $q_\Gamma > d - 1$ , such that  $\text{dom}(\mathcal{A}_u^{q, q_\Gamma}) = W^{1, q, q_\Gamma}$ . The following lemma summarizes useful embeddings for the corresponding function spaces.

**Lemma 2.4.** *If  $\text{dom}(\mathcal{A}_u^{q, q_\Gamma}) = W^{1, q, q_\Gamma}$ , then*

$$(1) \text{ for } \alpha \leq 1 - \frac{d}{q} \text{ and } \alpha_\Gamma \leq 1 - \frac{d-1}{q_\Gamma},$$

$$\text{dom}(\mathcal{A}_u^{q, q_\Gamma}) \hookrightarrow C^{\alpha, \alpha_\Gamma}, \quad (2.11)$$

$$(2) \text{ for any } 1 < r < \infty,$$

$$\text{MR}_{q, q_\Gamma}^r \hookrightarrow C^0(J_T; X_{q, q_\Gamma}^r). \quad (2.12)$$

*If  $q > d$ ,  $q_\Gamma > d - 1$ , and  $r > \max(\frac{2q}{q-d}, \frac{2q_\Gamma}{q_\Gamma-d+1})$ , then*

$$X_{q, q_\Gamma}^r \hookrightarrow C^{\beta, \beta_\Gamma}, \quad (2.13)$$

*where  $0 < \beta \leq 1 - \frac{d}{q} - \frac{2}{r}$  and  $0 < \beta_\Gamma \leq 1 - \frac{d-1}{q_\Gamma} - \frac{2}{r}$ .*

$$(3) \text{ for } q > d, q_\Gamma > d-1, \text{ let } 0 < \delta < \min(\frac{q-d}{2q}, \frac{q_\Gamma-d+1}{2q_\Gamma}) \text{ and } r > \max(\frac{2q}{q-2\delta q-d}, \frac{2q_\Gamma}{q_\Gamma-2\delta q_\Gamma-d+1}),$$

*then*

$$\text{MR}_{q, q_\Gamma}^r \hookrightarrow C^\delta(J_T; C^{\gamma, \gamma_\Gamma}) \quad (2.14)$$

*with  $0 < \gamma \leq 1 - \frac{d}{q} - \frac{2}{r} - 2\delta$  and  $0 < \gamma_\Gamma \leq 1 - \frac{d-1}{q_\Gamma} - \frac{2}{r} - 2\delta$ . In particular, the embedding*

$$\text{MR}_{q, q_\Gamma}^r \hookrightarrow C^0(J_T; C^{0,0}) \quad (2.15)$$

*is compact.*

*Proof.* Note that  $\Omega_+, \Omega_-$  and  $\Gamma$  are sufficiently regular for embedding and interpolation results to work “as usual”, i.e. as in the whole space. The first embedding (2.11) is standard, cf. e.g. [37, 2.8.1(c)]. For embedding (2.12), cf. [1, Section III.4.10]. Embedding (2.13) follows by definition of  $X_{q, q_\Gamma}^r$ , combining e.g. the interpolation result [37, p. 186, (14)] and the embedding [37, 2.8.1]. From [13, Lemma 3.4(b)], it follows that

$$\text{MR}_{q, q_\Gamma}^r \hookrightarrow C^\delta(J_T; (W^{-1, q, q_\Gamma}, W^{1, q, q_\Gamma})_{\theta, 1})$$

with  $0 < \theta \leq 1 - \frac{1}{r} - \delta$ . Embedding (2.14) then follows again by combining [37, p. 186, (14)] and [37, 2.8.1].  $\square$

We conclude this section with an assumption on the dependence of  $\mathcal{F}$  on  $u$  in (2.10).

**Assumption 2.5.** *Given  $1 < r < \infty$  and  $q, q_\Gamma > 2$ , the function  $\mathcal{F}: X_{q, q_\Gamma}^r \rightarrow W^{-1, q, q_\Gamma}$  is locally Lipschitz in the sense that for all  $\tilde{L} > 0$ , there exists a function  $\phi_{\tilde{L}} \in L^r(J_T; \mathbb{R})$  such that for all  $u_1, u_2 \in X_{q, q_\Gamma}^r$  with  $\|u_1\|_{X_{q, q_\Gamma}^r}, \|u_2\|_{X_{q, q_\Gamma}^r} \leq \tilde{L}$ ,*

$$\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|_{W^{-1, q, q_\Gamma}} \leq \phi_{\tilde{L}}(t) \|u_1 - u_2\|_{X_{q, q_\Gamma}^r}.$$

## 3. GLOBAL EXISTENCE AND UNIQUENESS

The main result of this section is global existence and uniqueness of solutions of (2.10). For local well-posedness, it is sufficient that  $\mathcal{F}$  satisfies Assumption 2.5. For global existence, we require that  $\mathcal{F}$  preserves a maximum principle for (2.10). This assumption is fairly general but remains vague. Examples for suitable  $\mathcal{F}$  are given in Corollary 3.6. Of course, the case  $\mathcal{F} = 0$  is included.

**Theorem 3.1.** *Let  $T > 0$ . Then there exist  $q > d$ ,  $q_\Gamma > d - 1$  such that for all  $r > \max(\frac{2q}{q-d}, \frac{2q_\Gamma}{q_\Gamma-d+1})$ ,  $u^0 \in X_{q,q_\Gamma}^r$  and  $\mathcal{F}$  satisfying Assumption 2.5 and preserving a maximum principle, there is a unique global solution*

$$u \in W^{1,r}(J_T; W^{-1,q,q_\Gamma}) \cap L^r(J_T; W^{1,q,q_\Gamma})$$

of (2.10). In particular, the solution is Hölder continuous in time and space,

$$u \in C^\delta(J_T; C^{\gamma,\gamma_\Gamma}),$$

with  $\delta, \gamma, \gamma_\Gamma$  as in Lemma 2.4.

The strategy of the proof is to use non-autonomous maximal regularity of the operators  $\mathcal{A}_{u(t)}^{q,q_\Gamma}$  combined with Schaefer's fixed point theorem and to obtain a priori bounds by a maximum principle. The proof is divided into four steps:

- (1) provisional reduction to bounded coefficients,
- (2) preliminary results on the linearized non-autonomous problem,
- (3) maximum principle,
- (4) Schaefer argument and proof of the theorem.

(1) **Provisional reduction to bounded coefficients.** By Lemma 2.4,  $u^0 \in C^{\beta,\beta_\Gamma} \subset C^{0,0}$ . Let  $-\infty < l_0 \leq L_0 < +\infty$  be such that  $u^0 \in C_{l_0}^{L_0}$ . Define

$$[f]_l^L(x) := \begin{cases} L, & f(x) \geq L, \\ l, & f(x) \leq l, \\ f(x), & \text{otherwise.} \end{cases}$$

and let  $L := L_0 + 1$ ,  $l = l_0/2$ . Instead of the coefficient functions  $k$  and  $m$ , we consider

$$k_l^L(\cdot, u(\cdot)) = k(\cdot, [u]_l^L(\cdot)) \quad \text{and} \quad m_l^L(\cdot, u(\cdot)) = m(\cdot, [u]_l^L(\cdot)) \quad (3.1)$$

in the following. In Step (4) below, it is shown that  $k_l^L = k$  and  $m_l^L = m$  along the orbits of  $u^0$ , thus concluding the proof of the theorem. Clearly, if  $k, m$  satisfy Assumption 2.2, then also  $k_l^L, m_l^L$  satisfy Assumption 2.2. In particular, the bounds in 2.2(1) and 2.2(2) hold uniformly in  $u \in C^{0,0}$  for  $k_l^L, m_l^L$ .

(2) **Preliminary results on the linearized non-autonomous problem.** In this step of the proof and in Step (3), using Step (1), we assume additionally that all coefficient functions are such that the bounds in 2.2(1) and 2.2(2) hold uniformly in  $u \in C^{0,0}$ .

**Lemma 3.2.** *There exist  $q > d$  and  $q_\Gamma > d - 1$  such that for any  $u \in C^0(J_T; C^{0,0})$ , for all  $t \in \overline{J_T}$ , for any  $\lambda > 0$ , the operator  $\mathcal{A}_{u(t)}^{q,q_\Gamma} + \lambda$  is an isomorphism*

$$\mathcal{A}_{u(t)}^{q,q_\Gamma} + \lambda: W^{1,q,q_\Gamma} \rightarrow W^{-1,q,q_\Gamma}. \quad (3.2)$$

*Proof.* First note that  $\mathcal{A}_{u(t)}^{q,q_\Gamma}: \text{dom}(\mathcal{A}_{u(t)}^{q,q_\Gamma}) \rightarrow W^{1,q,q_\Gamma}$  is well-defined for all  $t \in \overline{J_T}$ .

Consider the case  $d = 2$ . By the Lax-Milgram theorem, the claim holds for  $q = q_\Gamma = 2$ . By Sneiberg's theorem [34], the isomorphism property extrapolates to a neighbourhood of  $W^{1,2,2}$  in the complex interpolation scale  $[W^{1,p,p_\Gamma}, W^{1,p',p'_\Gamma}]_{1/p} = W^{1,2,2_\Gamma}$ ,  $1 < p, p_\Gamma < \infty$ , see [23].

If  $d = 3$ , then Assumption 2.2(4) holds. If  $\kappa_\pm \equiv 1$ , then  $k_\pm = \varkappa_\pm$  is independent of



$u$  and then by [24, Lemma 6.5], there is a  $q > 3$  such that the isomorphism property  $\mathcal{L}_{u(t),\pm} + \lambda: W^{1,q}(\Omega_{\pm}) \rightarrow W^{-1,q}(\Omega_{\pm})$  holds true. Using the same extrapolation argument as in the case  $d = 2$ , there exists a  $q_{\Gamma} > 2$  such that  $\mathcal{L}_{u(t),\Gamma} + \lambda: W^{1,q_{\Gamma}}(\Gamma) \rightarrow W^{-1,q_{\Gamma}}(\Gamma)$  is an isomorphism. In [10, Theorem 6.3] it was shown that the domains of  $\mathcal{L}_{u(t),\pm}, \mathcal{L}_{u(t),\Gamma}$  remain unchanged by a scalar multiplicative perturbation  $\kappa_{\pm} \in C^0(\Omega_{\pm})$  that is positively bounded from below. This proves the result for the operators  $\mathcal{L}_{u(t)}$ ,  $t \in \overline{J_T}$ . By relative boundedness of  $\mathcal{M}_{u(t)}$ , [9, Lemma 3.4], the domains of  $\mathcal{L}_{u(t)} + \lambda$  and  $\mathcal{A}_{u(t)} + \lambda$  coincide. This proves the claim.  $\square$

**Lemma 3.3.** *Let  $2 \leq q, q_{\Gamma} < \infty$ ,  $1 < r < \infty$  and let  $u \in C^0(J_T; C^{0,0})$ . Then for all  $t \in \overline{J_T}$ ,  $\mathcal{A}_{u(t)}^{q,q_{\Gamma}}$  has maximal  $L^r(J_T; W^{-1,q,q_{\Gamma}})$ -regularity.*

*Proof.* The result was shown in [9] if  $\Gamma$  is flat. It remains to check the maximal regularity of the Neumann operator  $\mathcal{L}_{u(t),\Gamma}$  on  $C^1$ -manifolds. This follows from maximal regularity for flat domains [23], using the usual localization methods, i.e. exploiting that the property of maximal regularity is preserved under perturbations that occur when locally flattening the domain and straightening the boundary with respect to a sufficiently fine covering and a corresponding partition of unity, see [7] for the general strategy and [8] for this argument in a similar context.  $\square$

**Lemma 3.4.** *Let  $w \in C^0(J_T; C^{0,0})$ ,  $q, q_{\Gamma}$  as in Lemma 3.2. Then for every  $r, u^0 \in X_{q,q_{\Gamma}}^r$  as in Theorem 3.1 and  $f \in L^r(J_T; W^{-1,q,q_{\Gamma}})$ , there exists a unique global solution  $v \in \text{MR}_{q,q_{\Gamma}}^r$  of*

$$\begin{aligned} \dot{v}(t) + \mathcal{A}_{w(t)}^{q,q_{\Gamma}} v(t) &= f(t), \quad \text{in } W^{-1,q,q_{\Gamma}}, \\ v(0) &= u^0, \end{aligned} \quad (3.3)$$

and the solution operator

$$(\partial_t + \mathcal{A}_{w(\cdot)}^{q,q_{\Gamma}})^{-1}: (f, u^0) \in L^r(J_T; W^{-1,q,q_{\Gamma}}) \times X_{q,q_{\Gamma}}^r \mapsto v \in \text{MR}_{q,q_{\Gamma}}^r \quad (3.4)$$

is continuous with Lipschitz dependence on  $w \in C^0(J_T; C^{0,0})$ .

*Proof.* For two Banach spaces  $X, Y$ , let  $\mathcal{B}(X, Y)$  denote the space of bounded linear operators  $B: X \rightarrow Y$ . By continuity of  $w$  and  $k_l^L$  and by Lemma 3.2, the map  $J_T \ni t \mapsto \mathcal{A}_w^{q,q_{\Gamma}}(t) \in \mathcal{B}(W^{1,q,q_{\Gamma}}, W^{-1,q,q_{\Gamma}})$  is uniformly continuous. By Lemma 3.4, for all  $t \in \overline{J_T}$ ,  $\mathcal{A}_w^{q,q_{\Gamma}}(t)$  has maximal  $L^r(J_T; W^{-1,q,q_{\Gamma}})$ -regularity, so existence and continuity of the solution operator follow from [32, Theorem 2.5].

By Assumption 2.2(3), given  $w_1, w_2 \in C^{0,0}$ , we obtain

$$\|\mathcal{A}_{w_1}^{q,q_{\Gamma}} - \mathcal{A}_{w_2}^{q,q_{\Gamma}}\|_{\mathcal{B}(W^{1,q,q_{\Gamma}}, W^{-1,q,q_{\Gamma}})} \leq C \|w_1 - w_2\|_{L^{\infty}}, \quad (3.5)$$

with  $C > 0$  independent of  $w_1, w_2 \in C^{0,0}$  and thus the dependence  $C^0(J_T; C^{0,0}) \ni w \mapsto \partial_t + \mathcal{A}_{w(\cdot)}^{q,q_{\Gamma}} \in \mathcal{B}(\text{MR}_{q,q_{\Gamma}}^r, L^r(J_T; W^{-1,q,q_{\Gamma}}) \times X_{q,q_{\Gamma}}^r)$  is Lipschitz, and the dependence on  $w$  of the inverse of the non-autonomous operator  $(\partial_t + \mathcal{A}_{w(\cdot)}^{q,q_{\Gamma}})^{-1}$  is Lipschitz as well.  $\square$

**(3) Maximum principle.** In this step, we prove uniform  $L^{\infty}$ -bounds on  $u$  from above and below. With respect to standard results for the bulk problems, cf. Corollary 5.1, the point is to show that the nonlinear bulk-interface interaction terms derived from a generalized gradient structure preserve this property.

**Lemma 3.5.** *(Bulk-Interface Maximum Principle) Let  $r, q, q_{\Gamma}$  as in Theorem 3.1,  $\mathcal{F} \equiv 0$  and  $u^0 \in X_{q,q_{\Gamma}}^r$  with  $u^0 \in C_l^L$  for some  $-\infty < l \leq L < +\infty$ . Assume that  $u \in \text{MR}_{q,q_{\Gamma}}^r$  is a solution of (2.10). Then for all  $t \in \overline{J_T}$ ,  $u(t) \in C_l^L$ .*

*Proof.* Define  $\zeta_l(t) = [(u(t) - l)^-]$  and  $\zeta^L(t) = [(L - u(t))^-]$ , where

$$[f^-](x) := \begin{cases} 0, & f(x) \geq 0, \\ -f(x), & f(x) < 0. \end{cases}$$

Since  $[\cdot]^-$  is Lipschitz and  $r, q, q_\Gamma \geq 2$ ,  $\zeta_l, \zeta^L \in L^r(J_T; W^{1,q,q_\Gamma}) \hookrightarrow L^{r'}(J_T; W^{1,q',q'_\Gamma})$  with

$$\nabla \zeta^L(t, x) = \begin{cases} 0, & u(t, x) \leq L, \\ \nabla u(t, x), & u(t, x) > L, \end{cases}$$

and  $\zeta_l(0) = \zeta^L(0) \equiv 0$ . For all  $s \in \overline{J_T}$ , testing (2.10) with  $\zeta^L$  in space and time gives  $\zeta^L \equiv 0$  as

$$\int_0^s \dot{u}(t)(\zeta^L(t)) \, dt = \frac{1}{2} \|\zeta^L(s)\|_{L^{2,2}}^2 \geq 0$$

and

$$\int_0^s \mathcal{A}_{u(t)}^{q,q_\Gamma} u(t)(\zeta^L(t)) \, dt = \int_0^s \mathfrak{l}_{u(t)}(u(t), \zeta^L(t)) + \mathfrak{m}_{u(t)}(u(t), \zeta^L(t)) \, dt \geq 0. \quad (3.6)$$

To show the estimate from below in (3.6), note that

$$\int_0^s \mathfrak{l}_{u(t)}(u(t), \zeta^L(t)) \, dt = \int_0^s \mathfrak{l}_{u(t)}(\zeta^L(t), \zeta^L(t)) \, dt \geq 0$$

as  $k$  is bounded below by  $\underline{k}$  and that

$$\begin{aligned} \int_0^s \mathfrak{m}_{u(t)}(u(t), \zeta^L(t)) \, dt &= \int_0^s \int_\Gamma m_+(u)(u_+ - u_\Gamma)(\zeta_+^L - \zeta_\Gamma^L)(t) \\ &\quad + m_-(u)(u_- - u_\Gamma)(\zeta_-^L - \zeta_\Gamma^L)(t) \\ &\quad + m_\Gamma(u)(u_+ - u_-)(\zeta_+^L - \zeta_-^L)(t) \, d\mathcal{H}_{d-1} \, dt, \end{aligned} \quad (3.7)$$

where  $m$  is bounded below by  $\underline{m}$  and where

$$\begin{aligned} \int_\Gamma (u_+ - u_\Gamma)(\zeta_+^L - \zeta_\Gamma^L)(t) \, d\mathcal{H}_{d-1} &= \int_{\{x \in \Gamma: u_+(x) > L > u_\Gamma(x)\}} (u_+ - u_\Gamma)(u_+ - L)(t) \, d\mathcal{H}_{d-1} \\ &\quad + \int_{\{x \in \Gamma: u_+(x) < L < u_\Gamma(x)\}} (u_+ - u_\Gamma)(L - u_\Gamma)(t) \, d\mathcal{H}_{d-1} \\ &\quad + \int_{\{x \in \Gamma: u_+(x), u_\Gamma(x) > L\}} (u_+ - u_\Gamma)(u_+ - u_\Gamma)(t) \, d\mathcal{H}_{d-1} \geq 0, \end{aligned}$$

and non-negativity of the remaining terms on the right-hand-side of (3.7) follows analogously. The proof of the lower bound, i.e.  $\zeta^l \equiv 0$  follows analogously by testing (2.10) with  $\zeta^l$ .  $\square$

If  $\mathcal{F} \neq 0$ , Theorem 3.1 still requires that  $\mathcal{F}$  preserves a maximum principle. A particular example is given by terms of Allen-Cahn-type, treated in the following corollary.

**Corollary 3.6.** *Let  $\mathcal{F}$  satisfy Assumption 2.5 and let all the components  $\varphi$  of  $\mathcal{F}$ , e.g.  $\varphi = f_+, g_-, \dots$  in (1.1)–(1.3) be independent of  $x \in \Omega_+, \Omega_-$ ,  $y \in \Gamma$ , respectively. Assume that  $\varphi$  are continuously differentiable in  $u$  and that  $g_\pm$  depend only on  $u_\pm$ , respectively, whereas  $f_\Gamma$  depends only on  $u_\Gamma$ . Assume that all  $\varphi$  satisfy the dissipativity condition*

$$\liminf_{|v| \rightarrow \infty} -\varphi'(v) > 0. \quad (3.8)$$

*Then, under the assumptions of the maximum principle Lemma 3.5, given a solution  $u \in \text{MR}_{q,q_\Gamma}^r$  of (2.10), there are constants  $-\infty < l_f \leq L_f < +\infty$ , such that for all  $t \in \overline{J_T}$ ,  $u(t) \in C_{l_f}^{L_f}$ .*

*Proof.* Condition (3.8) guarantees that for every component  $\varphi$ , there exist constants  $-\infty < l_\varphi \leq L_\varphi < +\infty$  such that  $\varphi(v) > 0$  for all  $v < l_\varphi$  and  $\varphi(v) < 0$  for all  $v > L_\varphi$ . Let  $l_f := \min_\varphi(l_\varphi)$  and  $L_f := \max_\varphi(L_\varphi)$ . In the choice of test functions  $\zeta_l, \zeta^L$  in the proof of Lemma 3.5, replace  $l, L$  by  $l_f, L_f$ . It is then straightforward to check that for all  $s \in \overline{J_T}$ ,

$\int_0^s \mathcal{F}(u(t))(\zeta^{L_f}(t)) dt \leq 0$  and that  $\int_0^s \mathcal{F}(u(t))(\zeta_{l_f}(t)) dt \geq 0$ . Combined with the calculations in the proof of Lemma 3.5, this proves the claim.  $\square$

(4) **Schaefer argument and proof of Theorem 3.1.** Let  $q, q_\Gamma$  be given by Lemma 3.2 and let  $r$  and  $u^0 \in X_{q, q_\Gamma}^r$  be given as in Theorem 3.1. By embedding (2.13),  $u^0 \in C_l^L$  for some  $-\infty < l \leq L < +\infty$ . In the following, let

$$C_{u^0}^0(J_T; C^{0,0}) := \{u \in C^0(J_T; C^{0,0}) : u(0) = u^0\}. \quad (3.9)$$

Define

$$\mathcal{T} : C_{u^0}^0(J_T; C^{0,0}) \rightarrow C_{u^0}^0(J_T; C^{0,0})$$

by  $\mathcal{T}w = v \in \text{MR}_{q, q_\Gamma}^s$  the solution of (3.3) with  $v(0) = u^0$  given by Lemma 3.4. By embedding (2.15),  $\mathcal{T}$  is well-defined and compact. Moreover,  $\mathcal{T}$  is Lipschitz continuous by Lemma 3.4 and a fixed point of  $\mathcal{T}$  would solve (2.10). To obtain existence of a fixed point by Schaefer's Theorem [16, Theorem 9.2.4], it suffices to show that the Schaefer set

$$S := \{u \in C_{u^0}^0(J_T; C^{0,0}) : u = \lambda \mathcal{T}(u) \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. If  $u_\lambda = \lambda \mathcal{T}(u_\lambda)$  for some  $0 < \lambda \leq 1$ , then by definition of  $\mathcal{T}$ ,  $u_\lambda \in \text{MR}_{q, q_\Gamma}^s$  and  $u_\lambda$  satisfies (2.10) with initial value  $u_\lambda(0) = \lambda u^0$  and right-hand-side  $\lambda \mathcal{F}(u_\lambda)$ . Thus, if  $\mathcal{F} \equiv 0$  or if  $\mathcal{F}$  is as in Corollary 3.6 or of a different form that uniformly bounds solutions, then  $S$  is bounded. By the Lipschitz property (3.5), all conditions for [31, Theorem 3.1] are satisfied, implying uniqueness of the solution  $u$ . In addition, the maximum principle shows that in fact  $k_l^L = k$  and  $m_l^L = m$  along orbits of  $u^0$ , justifying step (1) a posteriori with possible adjustments to the choice of  $l$  and  $L$  by Corollary 3.6, and concluding the proof of Theorem 3.1.

#### 4. EXPONENTIAL DECAY TO EQUILIBRIUM AND STABILITY

In the previous section, in some sense, the point was to show that the interaction of bulk and interface is sufficiently weak not to disturb the well-posedness of the Neumann problems on bulks and interface. Here, the point is to show that the interaction is sufficiently strong for forcing the system into the uniform equilibrium given by

$$u^\infty = \frac{1}{V} \left( \int_{\Omega_+} u_+^0(x) dx + \int_{\Omega_-} u_-^0(x) dx + \int_\Gamma u_\Gamma^0(y) d\mathcal{H}_{d-1} \right)$$

associated to  $u^0$ , where

$$V = |\Omega_+| + |\Omega_-| + |\Gamma|_{\mathcal{H}_{d-1}}.$$

By a slight abuse of notation,  $u^\infty$  also denotes the constant vector function  $u^\infty = u^\infty(1, 1, 1) \in C^{0,0}$ .

**Theorem 4.1.** *Under the assumptions of Theorem 3.1 with  $\mathcal{F} \equiv 0$ , given  $u^0 \in X_{q, q_\Gamma}^r$ , the solution  $u$  converges to  $u^\infty$  at an exponential rate, in the sense that there is a  $\delta > 0$  depending only on  $u^0, k, m, \Omega$  and  $\Gamma$ , such that for all  $s \geq 0$ ,*

$$\|u(s) - u^\infty\|_{L^{2,2}} \leq e^{-\delta s} \|u^0 - u^\infty\|_{L^{2,2}}. \quad (4.1)$$

*Proof.* Since for every solution  $u \in \text{MR}_{q, q_\Gamma}^r$  and  $T > 0$ ,  $\mathbf{a}_{u(s)}(u(s), u^\infty) = 0$ , testing (2.10) with  $u - u^\infty$  shows the energy balance

$$\|u(s) - u^\infty\|_{L^{2,2}}^2 + \int_0^s \mathbf{a}_{u(t)}(u(t), u(t)) dt = \|u^0 - u^\infty\|_{L^{2,2}}^2, \quad (4.2)$$

for all  $s > 0$ . By Lemma 3.5 and Assumption 2.2,

$$\mathbf{l}_{u(t)}(u(t), u(t)) \geq C \|\nabla u(t)\|_{L^{2,2}}^2, \text{ and}$$

$$\mathbf{m}_{u(t)}(u(t), u(t)) \geq \underline{m} \left( \int_\Gamma (u_+ - u_\Gamma)^2(t) + \int_\Gamma (u_- - u_\Gamma)^2(t) + \int_\Gamma (u_+ - u_-)^2(t) dy \right).$$

Hence, with the following Poincaré-type inequality, the claim follows directly from Gronwall's inequality.  $\square$

**Lemma 4.2.** (*Bulk-Interface Poincaré Inequality*) *Let  $u \in W^{1,2,2}$  and  $u^\infty$  the equilibrium associated to  $u$ . Then there is a constant  $C > 0$ , independent of  $u$ , such that*

$$\|u - u^\infty\|_{L^{2,2}}^2 \leq C(\|\nabla u\|_{L^{2,2}}^2 + \|u_+ - u_\Gamma\|_{L^2(\Gamma)}^2 + \|u_- - u_\Gamma\|_{L^2(\Gamma)}^2 + \|u_+ - u_-\|_{L^2(\Gamma)}^2). \quad (4.3)$$

*Proof.* For any  $u \in L^{1,1}$ , let in the following  $\bar{u}_+ := \frac{1}{|\Omega_+|} \int_{\Omega_+} u_+$ ,  $\bar{u}_- := \frac{1}{|\Omega_-|} \int_{\Omega_-} u_-$  and  $\bar{u}_\Gamma := \frac{1}{|\Gamma|} \int_\Gamma u_\Gamma$  and let  $\bar{u} = (\bar{u}_+, \bar{u}_-, \bar{u}_\Gamma) \in \mathbb{R}^3$ . To prove (4.3), we use the following two (standard) versions of Poincaré's inequality [4, Theorem 1 and Corollary 3]. For all  $u_+ \in W^{1,p}(\Omega_+)$ ,

(1) there is a constant  $\bar{C}_+ > 0$ , such that

$$\|u_+ - \bar{u}_+\|_{L^2(\Omega_+)}^2 \leq \bar{C}_+ \|\nabla u_+\|_{L^2(\Omega_+)}^2, \quad \text{and}, \quad (4.4)$$

(2) there is a constant  $C_+^\Gamma > 0$ , such that

$$\|u_+\|_{L^2(\Omega_+)}^2 \leq C_+^\Gamma (\|\nabla u_+\|_{L^2(\Omega_+)}^2 + \frac{1}{|\Gamma|} \left| \int_\Gamma u_+ \right|^2). \quad (4.5)$$

Clearly, analogous statements hold for  $\Omega_-$  with constants  $\bar{C}_- > 0$  and  $C_-^\Gamma > 0$  and (4.4) holds for  $u_\Gamma$  on the manifold  $\Gamma$  with constant  $\bar{C}_\Gamma > 0$ . An elementary calculation shows that

$$\|u - u^\infty\|_{L^{2,2}}^2 = \|u - \bar{u}\|_{L^{2,2}}^2 - V(u^\infty)^2 + |\Omega_+|\bar{u}_+^2 + |\Omega_-|\bar{u}_-^2 + |\Gamma|\bar{u}_\Gamma^2.$$

Inserting  $Vu^\infty = |\Omega_+|\bar{u}_+ + |\Omega_-|\bar{u}_- + |\Gamma|\bar{u}_\Gamma$  gives

$$\|u - u^\infty\|_{L^{2,2}}^2 = \|u - \bar{u}\|_{L^{2,2}}^2 + \frac{|\Omega_+||\Omega_-|}{V}(\bar{u}_+ - \bar{u}_-)^2 + \frac{|\Omega_+||\Gamma|}{V}(\bar{u}_+ - \bar{u}_\Gamma)^2 + \frac{|\Omega_-||\Gamma|}{V}(\bar{u}_- - \bar{u}_\Gamma)^2. \quad (4.6)$$

By (4.4),  $\|u - \bar{u}\|_{L^{2,2}}^2 \leq (\bar{C}_+ + \bar{C}_- + \bar{C}_\Gamma) \|\nabla u\|_{L^{2,2}}^2$ , so it remains to estimate the last three terms in (4.6) by the right-hand-side in (4.3). By Hölder's inequality and by (4.5),

$$\begin{aligned} (\bar{u}_+ - \bar{u}_\Gamma)^2 &= \frac{1}{|\Omega_+|^2} \left( \int_{\Omega_+} u_+ - \bar{u}_\Gamma \right)^2 \leq \frac{1}{|\Omega_+|} \|u_+ - \bar{u}_\Gamma\|_{L^2(\Omega_+)}^2 \\ &\leq \frac{C_+^\Gamma}{|\Omega_+|} (\|\nabla u_+\|_{L^2(\Omega_+)}^2 + \frac{1}{|\Gamma|} \left| \int_\Gamma u_+ - \bar{u}_\Gamma \right|^2) \\ &\leq \frac{C_+^\Gamma}{|\Omega_+|} (\|\nabla u_+\|_{L^2(\Omega_+)}^2 + \|u_+ - u_\Gamma\|_{L^2(\Gamma)}^2). \end{aligned}$$

The term  $(\bar{u}_- - \bar{u}_\Gamma)^2$  can be estimated analogously. In order to estimate the last term  $(\bar{u}_+ - \bar{u}_-)^2$ , simply insert  $-\bar{u}_\Gamma + \bar{u}_\Gamma$  and use the previous estimates. With this strategy, it is clear that for (4.3) to hold, it is sufficient that two of the three coefficient functions  $m_+, m_-, m_\Gamma$  are positive, so not every pair of unknowns needs to interact across  $\Gamma$ . Note that it is also sufficient for two of these coefficients to be positive to guarantee the structure of the kernel of  $\mathbf{a}_u$  in (2.8). This concludes the proof of Lemma 4.2 and thus of Theorem 4.1.  $\square$

In addition to exponential stability of  $u^\infty$  within the sets of initial data with equal mass, Theorem 4.1 immediately implies stability of  $u^\infty$  in  $X_{q,q_\Gamma}^r$ :

**Corollary 4.3.** *For every  $v^\infty \in \mathbb{R}_+$ ,  $\varepsilon > 0$ , if  $u^0 \in X_{q,q_\Gamma}^r$  with  $\|u^0 - v^\infty\|_{L^{1,1}} < \varepsilon V$ , then  $|u^\infty - v^\infty| < \varepsilon$ .*

*Proof.* A direct calculation shows that

$$\begin{aligned} |u^\infty - v^\infty| &= \frac{1}{V} \left| \int_{\Omega_+} u_+^0(x) - v^\infty \, dx + \int_{\Omega_-} u_-^0(x) - v^\infty \, dx + \int_\Gamma u_\Gamma^0(y) - v^\infty \, dy \right| \\ &\leq \frac{1}{V} \|u^0 - v^\infty\|_{L^{1,1}}. \end{aligned}$$

□

## 5. EXTENSIONS AND CONCLUDING REMARKS

**5.1. Entropic gradient structure for heat transfer (Onsager model).** Originally, the system in (1.1)–(1.3) was motivated by non-equilibrium thermodynamical modeling of heat transfer and diffusion processes across interfaces, [29], [25], and based on the results in [20] and [27]. For example, in [27], it is shown that for *flat* interfaces  $\Gamma$ , the heat transfer *Onsager* or *gradient system* associated to

$$\dot{\theta} = \mathcal{K}(\theta) \mathcal{D}\mathcal{S}(\theta) \quad (5.1)$$

is represented by the set of equations

$$\begin{cases} \dot{\theta}_{\pm} + \frac{1}{c_{\pm}} \operatorname{div}(K_{\pm}(\theta_{\pm}) \nabla \frac{1}{\theta_{\pm}}) = 0, & \text{in } (0, T) \times \Omega_{\pm}, \\ (\frac{K_{\pm}(\theta_{\pm})}{c_{\pm}} \nabla \frac{1}{\theta_{\pm}}) \nu_{\pm} + M_{\pm}(\theta) (\frac{1}{\theta_{\pm}} - \frac{1}{\theta_{\Gamma}}) + M_{\Gamma}(\theta) (\frac{1}{\theta_{\pm}} - \frac{1}{\theta_{m_{\pm}}}) = 0, & \text{on } (0, T) \times \Gamma, \\ (K_{\pm}(\theta_{\pm}) \nabla \frac{1}{\theta_{\pm}}) \nu_{\pm} = 0, & \text{on } (0, T) \times \{\partial\Omega_{\pm} \setminus \Gamma\}, \end{cases} \quad (5.2)$$

on the bulk parts, and

$$\begin{cases} \dot{\theta}_{\Gamma} + \frac{1}{c_{\Gamma}} \operatorname{div}(K_{\Gamma}(\theta) \nabla \frac{1}{\theta_{\Gamma}}) - M_{+}(\theta) (\frac{1}{\theta_{+}} - \frac{1}{\theta_{\Gamma}}) - M_{-}(\theta) (\frac{1}{\theta_{-}} - \frac{1}{\theta_{\Gamma}}) = 0, & \text{in } (0, T) \times \Gamma, \\ (K_{\Gamma}(\theta) \nabla \frac{1}{\theta_{\Gamma}}) \nu_{\Gamma} = 0, & \text{on } (0, T) \times \partial\Gamma, \end{cases} \quad (5.3)$$

on the flat interface  $\Gamma$ , where  $c_{\pm}, c_{\Gamma} > 0$  are the specific heats of bulk and interface materials, respectively, and the coefficients  $K, M$  specify thermal conductivity within materials and across  $\Gamma$  in an entropic modelling. In (5.1),  $\mathcal{S}$  is the total entropy functional

$$\mathcal{S}(\theta) = \int_{\Omega_{+}} c_{+} \log \theta_{+} \, dx + \int_{\Omega_{-}} c_{-} \log \theta_{-} \, dx + \int_{\Gamma} c_{\Gamma} \log \theta_{\Gamma} \, dy,$$

and  $\mathcal{K}$  is the Onsager operator corresponding to the the dual dissipation potential

$$\begin{aligned} 2\Psi^{*}(\theta, \phi) &= 2\Psi_{+}^{*}(\theta_{+}, \phi_{+}) + 2\Psi_{-}^{*}(\theta_{-}, \phi_{-}) + 2\Psi_{\Gamma}^{*}(\operatorname{tr}_{\Gamma} \theta, \operatorname{tr}_{\Gamma} \phi) \\ &= \int_{\Omega_{+}} \nabla \frac{\phi_{+}}{c_{+}} \cdot K_{+}(\theta_{+}) \nabla \frac{\phi_{+}}{c_{+}} \, dx + \int_{\Omega_{-}} \nabla \frac{\phi_{-}}{c_{-}} \cdot K_{-}(\theta_{-}) \nabla \frac{\phi_{-}}{c_{-}} \, dx \\ &\quad + \int_{\Gamma} \nabla_{\Gamma} \frac{\phi_{\Gamma}}{c_{\Gamma}} \cdot K_{\Gamma}(\operatorname{tr}_{\Gamma} \theta) \nabla_{\Gamma} \frac{\phi_{\Gamma}}{c_{\Gamma}} \, dy + \int_{\Gamma} M_{\Gamma}(\operatorname{tr}_{\Gamma} \theta) \left( \frac{\operatorname{tr}_{\Gamma} \phi_{+}}{\operatorname{tr}_{\Gamma} c_{+}} - \frac{\operatorname{tr}_{\Gamma} \phi_{-}}{\operatorname{tr}_{\Gamma} c_{-}} \right)^2 \, dy \\ &\quad + \int_{\Gamma} M_{+}(\operatorname{tr}_{\Gamma} \theta) \left( \frac{\operatorname{tr}_{\Gamma} \phi_{+}}{\operatorname{tr}_{\Gamma} c_{+}} - \frac{\operatorname{tr}_{\Gamma} \phi_{\Gamma}}{\operatorname{tr}_{\Gamma} c_{\Gamma}} \right)^2 + M_{-}(\operatorname{tr}_{\Gamma} \theta) \left( \frac{\operatorname{tr}_{\Gamma} \phi_{-}}{\operatorname{tr}_{\Gamma} c_{-}} - \frac{\operatorname{tr}_{\Gamma} \phi_{\Gamma}}{\operatorname{tr}_{\Gamma} c_{\Gamma}} \right)^2 \, dy. \end{aligned} \quad (5.4)$$

Formally, (5.2), (5.3) are equivalent to (1.1)–(1.3) by differentiating  $\nabla \frac{1}{\theta}$  to  $-\frac{1}{\theta^2} \nabla \theta$  and writing  $\frac{1}{\theta_{\Gamma} \theta_{\pm}} (\theta_{+} - \theta_{\Gamma})$  instead of  $(\frac{1}{\theta_{\Gamma}} - \frac{1}{\theta_{\pm}})$ , for every term of this kind. The coefficients  $K$  and  $k$  and  $M$  and  $m$  are then related via  $m_{\pm}(\operatorname{tr}_{\Gamma} \theta) = \frac{M_{\pm}(\operatorname{tr}_{\Gamma} \theta)}{\theta_{\Gamma} \operatorname{tr}_{\Gamma} \theta_{\pm}}$ ,  $m_{\Gamma}(\operatorname{tr}_{\Gamma} \theta) = \frac{M_{\Gamma}(\operatorname{tr}_{\Gamma} \theta)}{\operatorname{tr}_{\Gamma} \theta_{+} \operatorname{tr}_{\Gamma} \theta_{-}}$ ,  $k_{\pm}(\theta_{\pm}) = \frac{K_{\pm}(\theta_{\pm})}{\theta_{\pm}^2}$  and  $k_{\Gamma}(\operatorname{tr}_{\Gamma} \theta) = \frac{K_{\Gamma}(\operatorname{tr}_{\Gamma} \theta)}{\theta_{\Gamma}^2}$ .

It is straightforward to check that  $K, M$  satisfy Assumption 2.2 if and only if  $k, m$  satisfy Assumption 2.2. So if Assumption 2.2 on  $K, M$  is respected in an entropic modeling, well-posedness and exponential stability follow directly. In particular, the positivity of two components of  $M$  guarantees entropy production of the bulk-interface interaction.

Based on the previous analysis, we can retrieve information on the Onsager system given by  $\mathcal{S}$  and  $\Psi^{*}$ . Starting from positive initial values,  $l > 0$ , the regularity in Theorem 3.1 and the maximum principle a posteriori justify the equivalence of (5.2), (5.3) and (1.1)–(1.3) and the solution provides the gradient flow of  $\mathcal{S}$  with respect to the dual dissipation metric  $\Psi^{*}$ . The entropy  $\mathcal{S}(\theta(t))$  is well-defined along orbits and  $-\mathcal{S}$  provides a strict Lyapunov functional by the energy balance  $-\frac{d}{dt} \mathcal{S}(\theta(t)) + 2\Psi^{*}(\theta(t), \frac{c}{\theta(t)}) = 0$  and the fact that  $2\Psi^{*}(\theta(t), \frac{c}{\theta(t)}) = 0$  implies  $\mathbf{a}_{\theta(t)}(\theta(t), \theta(t)) = 0$  along the positive orbits of  $\theta$ . By Theorem

4.1, exponential stability holds in the sense that  $\|c\theta(t) - c\theta^\infty\|_{L^{2,2}} \leq e^{-\delta t} \|c\theta^0 - c\theta^\infty\|_{L^{2,2}}$  for some  $\delta > 0$ .

**5.2. Further remarks.** A direct corollary of Theorems 3.1 and 4.1 is the well-posedness of the porous medium and fast diffusion equation  $\dot{u} = \Delta u^\rho$  on a bounded Lipschitz domain with Neumann boundary conditions and  $\rho > 0$ . Although regularity and blow-up behaviour of the porous medium and fast diffusion equations in the whole space and with homogeneous Dirichlet boundary conditions is complex and well-studied, [38, 30], the following simple result for Neumann boundary conditions, based on the maximum principle, doesn't seem to be explicit in the literature, compare [5]. We consider equation (1.1) with  $\Gamma = \emptyset$  and call it (PME).

**Corollary 5.1.** *Consider (PME) with  $f_+, h_+ = 0$  and  $k_+(u_+) = \kappa_0 u_+^{\rho-1}$  for some  $\kappa_0 > 0$  and  $\rho \in \mathbb{R}$ . Then for every  $q > d$ ,  $r > \frac{2q}{q-d}$ , and positive  $u^0 \in (W^{1,q}(\Omega_+), W^{-1,q}(\Omega_+))_{1-\frac{1}{r},r}$ , there is a unique positive solution*

$$u \in W^{1,r}(J_T; W^{-1,q}(\Omega_+)) \cap L^r(J_T; W^{1,q}(\Omega_+))$$

of (PME) with  $\int_{\Omega_+} u(t) dx = \int_{\Omega_+} u^0 dx$  and

$$\|u(t) - \bar{u}\|_{L^2(\Omega_+)} \leq e^{-\delta t} \|u^0 - \bar{u}\|_{L^2(\Omega_+)}$$

for some  $\delta > 0$  and for all  $t \geq 0$ . Global existence and uniqueness of  $u$  extends to  $f_+, h_+ \neq 0$  if they satisfy Assumption 2.5 and preserve the maximum principle for the equation.

The remaining remarks concern extensions of Theorem 3.1, mostly based on perturbation theory for maximal parabolic regularity.

**Remark 5.2.** *If the Lipschitz dependence of  $k, m$  and  $\mathcal{F}$  on  $u$  in Assumptions 2.2(3) and 2.5 is improved to  $C^n$ ,  $n \in \mathbb{N} \cup \{\infty, \omega\}$ , then the solution  $u$  in Theorem 3.1 gains time regularity by [31, Theorem 5.1], i.e. it follows that*

$$u \in C^m(J_T; X_{r,q,q\Gamma}) \cap C^{m+1-1/r}(J_T; W^{-1,q,q\Gamma}) \cap C^{n-1/r}(J_T; W^{1,q,q\Gamma})$$

and that  $u \in C^\infty(J_T; W^{1,q,q\Gamma})$  if  $n = \infty$  and  $u$  is real analytic on  $J_T$  if  $n = \omega$ .

**Remark 5.3.** *Clearly, the analysis above includes the simpler case of bulk-interface interaction with  $\Omega_- = \emptyset$ , without the variable  $u_-$  and with  $m_- = 0$ .*

**Remark 5.4.** *The coefficient functions  $k, m$  and external forces and inhomogeneous boundary conditions  $f, g, h$  may additionally depend on time. For example, Theorems 3.1 and 4.1 continue to hold if Assumption 2.2 holds uniformly in  $t \in (0, \infty)$  for  $k, m$  and  $t \mapsto \mathcal{A}_{u(t)} \in \mathcal{B}(W^{1,q,q\Gamma}, W^{-1,q,q\Gamma})$  is continuous for all  $u \in X_{r,q,q\Gamma}^r$  and if Assumption 2.5 holds and  $t \mapsto \mathcal{F}(t, u) \in W^{-1,q,q\Gamma}$  is measurable, cf. [31, Section 3].*

**Remark 5.5.** *The condition  $\varkappa_\pm \in C(\overline{\Omega}_\pm)^{3 \times 3}$  in Assumption 2.2(4) may be relaxed considerably, e.g. to hold only piecewise on layers. The only point is to guarantee the isomorphism property in Lemma 3.2 in the case  $d = 3$ . For a detailed discussion of necessary and sufficient conditions for this property, we refer to [11]. Note that for measurable, bounded and elliptic coefficients in general there are counterexamples, if non-smoothness of  $\varkappa_\pm$  and non-smoothness of  $\partial\Omega_\pm$  meet, [15].*

**Remark 5.6.** *The results in Theorem 3.1 extend to perturbations of  $\mathcal{A}_q$  by lower-order terms like transport terms  $b \cdot \nabla u_\pm$ ,  $b \in \mathbb{R}^d$ . In particular, with suitable regularity assumptions, the coefficients  $c_\pm : \Omega_\pm \rightarrow \mathbb{R}_+ \setminus \{0\}$  and  $c_\Gamma : \Gamma \rightarrow \mathbb{R}_+ \setminus \{0\}$  in Subsection 5.1 can be chosen to depend on the spatial variables.*

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## REFERENCES

- [1] H. Amann. *Linear and Quasilinear Parabolic Problems. Vol. I*, volume 89 of *Monographs in Mathematics*. Birkhäuser, Boston, 1995.
- [2] W. Arendt and R. Chill. Global existence for quasilinear diffusion equations in isotropic nondivergence form. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 9(3):523–539, 2010.
- [3] D. Bothe, M. Köhne, S. Maier, and J. Saal. Global strong solutions for a class of heterogeneous catalysis models. *J. Math. Anal. Appl.*, 445(1):677–709, 2017.
- [4] A. Boulkhémair and A. Chakib. On the uniform Poincaré inequality. *Comm. Partial Differential Equations*, 32(7-9):1439–1447, 2007.
- [5] Chainais-Hillairet, Claire, Jüngel, Ansgar, and Schuchnigg, Stefan. Entropy-dissipative discretization of nonlinear diffusion equations and discrete Beckner inequalities. *ESAIM: M2AN*, 50(1):135–162, 2016.
- [6] P. Colli and J. Sprekels. Optimal control of an Allen-Cahn equation with singular potentials and dynamic boundary condition. *SIAM J. Control Optim.*, 53(1):213–234, 2015.
- [7] R. Denk, M. Hieber, and J. Prüss. R-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 2003.
- [8] R. Denk, J. Prüss, and R. Zacher. Maximal  $L_p$ -regularity of parabolic problems with boundary dynamics of relaxation type. *J. Funct. Anal.*, 255(11):3149–3187, 2008.
- [9] K. Disser. Well-posedness for coupled bulk-interface diffusion with mixed boundary conditions. *Analysis*, 35(4):309–317, 2015.
- [10] K. Disser, H.-C. Kaiser, and J. Rehberg. Optimal Sobolev regularity for linear second-order divergence elliptic operators occurring in real-world problems. *SIAM J. Math. Anal.*, 47(3):1719–1746, 2015.
- [11] K. Disser, H.-C. Kaiser, and J. Rehberg. Optimal sobolev regularity for linear second-order divergence elliptic operators occurring in real-world problems. *SIAM J. Math. Anal.*, 47(3):1719–1746, 2015.
- [12] K. Disser, M. Meyries, and J. Rehberg. A unified framework for parabolic equations with mixed boundary conditions and diffusion on interfaces. *J. Math. Anal. Appl.*, 430:1102–1123, 2015.
- [13] K. Disser, J. Rehberg, and A. ter Elst. Hölder estimates for parabolic operators on domains with rough boundary. *WIAS Preprint 2097, Ann. Sc. Norm. Sup. Pisa, accepted*, 2015.
- [14] G. Dore. Maximal regularity in  $l^p$  spaces for an abstract Cauchy problem. *Adv. Differential Equations*, 5:293–322, 2000.
- [15] J. Elschner, J. Rehberg, and G. Schmidt. Optimal regularity for elliptic transmission problems including  $C^1$  interfaces. *Interfaces Free Bound.*, 9(2):233–252, 2007.
- [16] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
- [17] J. R. Fernández, P. Kalita, S. Migórski, M. C. Muñoz, and C. Núñez. Existence and Uniqueness Results for a Kinetic Model in Bulk-Surface Surfactant Dynamics. *SIAM J. Math. Anal.*, 48(5):3065–3089, 2016.
- [18] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [19] A. Glitzky. An electronic model for solar cells including active interfaces and energy resolved defect densities. *SIAM J. Math. Anal.*, 44(6):3874–3900, 2012.
- [20] A. Glitzky and A. Mielke. A gradient structure for systems coupling reaction-diffusion effects in bulk and interfaces. *Z. Angew. Math. Phys.*, 64(1):29–52, 2013.
- [21] A. Glitzky and A. Mielke. A gradient structure for systems coupling reaction-diffusion effects in bulk and interfaces. WIAS-Preprint 1603. *ZAMP*, 64:29–52, 2013.
- [22] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.
- [23] K. Gröger. A  $W^{1,p}$ -estimate for solutions to mixed boundary value problems for second order elliptic differential equations. *Math. Ann.*, 283(4):679–687, 1989.
- [24] R. Haller-Dintelmann and J. Rehberg. Maximal parabolic regularity for divergence operators including mixed boundary conditions. *J. Differential Equations*, 247(5):1354–1396, 2009.

- [25] S. Kjelstrup and D. Bedeaux. *Non-equilibrium thermodynamics of heterogeneous systems*, volume 16 of *Series on Advances in Statistical Mechanics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.
- [26] A. Mielke. Thermomechanical modeling of energy-reaction-diffusion systems, including bulk-interface interactions. *Discrete Contin. Dyn. Syst. Ser. S*, 6(2):479–499, 2013.
- [27] A. Mielke. Thermomechanical modeling of energy-reaction-diffusion systems, including bulk-interface interactions. *Discrete Contin. Dyn. Syst. Ser. S*, 6(2):479–499, 2013.
- [28] R. Nittka. Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains. *J. Differential Equations*, 251(4-5):860–880, 2011.
- [29] H. Öttinger. *Beyond Equilibrium Thermodynamics*. John Wiley, New Jersey, 2005.
- [30] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.
- [31] J. Prüss. Maximal regularity for evolution equations in  $L_p$ -spaces. *Conf. Semin. Mat. Univ. Bari*, (285):1–39 (2003), 2002.
- [32] J. Prüss and R. Schnaubelt. Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time. *J. Math. Anal. Appl.*, 256(2):405–430, 2001.
- [33] J. Prüss and G. Simonett. *Moving Interfaces and Quasilinear Parabolic Evolution Equations*, volume 105 of *Monographs in Mathematics*. Birkhäuser Basel, 2016.
- [34] I. J. Šneĭberg. Spectral properties of linear operators in interpolation families of Banach spaces. *Mat. Issled.*, 9(2(32)):214–229, 254–255, 1974.
- [35] J. Sprekels and H. Wu. A note on parabolic equation with nonlinear dynamical boundary condition. *Nonlinear Anal.*, 72(6):3028–3048, 2010.
- [36] Y. S. Touloukian, R. W. Powell, C. Y. Ho, and P. G. Klemens. *Thermophysical Properties of Matter - The TPRC Data Series. Volume 1. Thermal Conductivity – Metallic Elements and Alloys*. THERMOPHYSICAL AND ELECTRONIC PROPERTIES INFORMATION ANALYSIS CENTER, LAFAYETTE IN, 1970.
- [37] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. Johann Ambrosius Barth, Heidelberg, 1995.
- [38] J. L. Vázquez. *The porous medium equation*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007. Mathematical theory.

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